

BIMINIMAL PROPERLY IMMERSED SUBMANIFOLDS IN THE EUCLIDEAN SPACES

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ABSTRACT. We consider a *complete nonnegative biminimal* submanifold M (that is, a complete biminimal submanifold with $\lambda \geq 0$) in a Euclidean space \mathbb{E}^N . Assume that the immersion is *proper*, that is, the preimage of every compact set in \mathbb{E}^N is also compact in M . Then, we prove that M is minimal. From this result, we give an affirmative partial answer to Chen's conjecture. For the case of $\lambda < 0$, we construct examples of biminimal submanifolds and curves.

1. Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional $E(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 v_g$, for smooth maps $\phi : (M^n, g) \rightarrow (\tilde{M}^N, \langle \cdot, \cdot \rangle)$.

On the other hand, in 1981, J. Eells and L. Lemaire [8] proposed the problem to consider the *polyharmonic maps of order k* (k -harmonic maps): they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi) v_g, \quad (k = 1, 2, \dots),$$

where $e_k(\phi) = \frac{1}{2} \|(d + d^*)^k \phi\|^2$ for smooth maps $\phi : (M^n, g) \rightarrow (\tilde{M}^N, \langle \cdot, \cdot \rangle)$. G.Y. Jiang [11] studied the first and second variational formulas of the bi-energy E_2 , and critical maps of E_2 are called *biharmonic maps* (*2-harmonic maps*). There have been extensive studies on biharmonic maps. The Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \sum_{i=1}^n R^{\tilde{M}}(\tau(\phi), d\phi(e_i)) d\phi(e_i) = 0,$$

where $\Delta^\phi := \sum_{i=1}^n (\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i} e_i}^\phi)$, $\tau(\phi) := \text{trace} \nabla d\phi$, $R^{\tilde{M}}$ and $\{e_i\}$ are the rough Laplacian, the tension field of ϕ , the Riemannian curvature of \tilde{M} i.e., $R^{\tilde{M}}(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ for any vector field X, Y and Z on \tilde{M} , and a local orthonormal frame field of M , respectively. If an isometric immersion $\phi : (M, g) \rightarrow (\tilde{M}, \langle \cdot, \cdot \rangle)$ is biharmonic, then M is called *biharmonic submanifold*.

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For biharmonic submanifolds, there is an interesting problem, namely, Chen's Conjecture (cf. [2]):

Conjecture 1. *Any biharmonic submanifold M in \mathbb{E}^N is minimal.*

There are many affirmative partial answers to Conjecture 1 (cf. [2, 3, 4, 6, 7, 9]). In particular, there are some complete affirmative answers if M is one of the following: (a) a curve [7], (b) a surface in \mathbb{E}^3 [2], (c) a hypersurface in \mathbb{E}^4 [6, 9].

On the other hand, since there is no assumption of *completeness* for submanifolds in Conjecture 1, in a sense it is a problem in *local* differential geometry. Recently, we reformulated Conjecture 1 into a problem in *global* differential geometry as the following (cf. [1, 14, 15]):

Conjecture 2. *Any complete biharmonic submanifold in \mathbb{E}^N is minimal.*

An immersed submanifold M in \mathbb{E}^N is said to be *properly immersed* if the immersion $M \rightarrow \mathbb{E}^N$ is a proper map. K. Akutagawa and the author showed that biharmonic properly immersed submanifold in the Euclidean space is minimal [1]. Here, we remark that the properness of the immersion implies the completeness of (M, g) .

Recently, E. Loubeau and S. Montaldo introduced *biminimal immersion* :

Definition 1.1 ([13]). An immersion $\phi : (M^n, g) \rightarrow (\tilde{M}^N, \langle \cdot, \cdot \rangle)$, $n \leq N$ is called *biminimal* if it is a critical point of the functional

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}$$

for any smooth variation of the map $\phi_t (-\epsilon < t < \epsilon)$, $\phi_0 = \phi$ such that $V = \frac{d\phi_t}{dt} \Big|_{t=0}$ is normal to $\phi(M)$.

The Euler-Lagrange equation for biminimal immersion is

$$[\tau_2(\phi)]^\perp + \lambda[\tau(\phi)]^\perp = 0,$$

where, $[\cdot]^\perp$ denotes the normal component of $[\cdot]$. We call an immersion *free biminimal* if it is biminimal condition for $\lambda = 0$. (It is sometimes called that biminimal is λ -biminimal and free biminimal is biminimal, respectively). If $\phi : (M, g) \rightarrow (\tilde{M}, \langle \cdot, \cdot \rangle)$ is an isometric immersion, then the biminimal condition is

$$(1) \quad [-\triangle^\phi \mathbf{H} - \sum_{i=1}^n R^{\tilde{M}}(\mathbf{H}, d\phi(e_i))d\phi(e_i)]^\perp + \lambda \mathbf{H} = 0,$$

for some $\lambda \in \mathbb{R}$. If an isometric immersion ϕ is biminimal, then M is called *biminimal submanifold*.

Remark 1.2. we remark that every biharmonic submanifold is free biminimal one.

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. In section 3, we show non-negative biminimal properly immersed submanifold (that is, a biminimal properly immersed submanifold with $\lambda \geq 0$) in the Euclidean space is minimal and get an affirmative partial answer to Chen's conjecture. In section 4, we construct examples of biminimal submanifolds and curves for the case of $\lambda < 0$.

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2. Preliminaries

Let M be an n -dimensional immersed submanifold in \mathbb{E}^N , $\mathbf{x} : M \rightarrow \mathbb{E}^N$ its immersion and g its induced Riemannian metric. For simplicity, we often identify M with its immersed image $\mathbf{x}(M)$ in every local arguments. Let ∇ and D denote respectively the Levi-Civita connections of (M, g) and $\mathbb{E}^N = (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$. For any vector fields $X, Y \in \mathfrak{X}(M)$, the Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y),$$

where h stands for the second fundamental form of M in \mathbb{E}^N . For any normal vector field ξ , the Weingarten map A_ξ with respect to ξ is given by

$$D_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where ∇^\perp stands for the normal connection of the normal bundle of M in \mathbb{E}^N . It is well known that h and A are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

For any $x \in M$, let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_N\}$ be an orthonormal basis of \mathbb{E}^N at x such that $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x M$. Then, h is decomposed as at x

$$h(X, Y) = \sum_{\alpha=n+1}^N h_\alpha(X, Y) e_\alpha.$$

The mean curvature vector \mathbf{H} of M at x is also given by

$$\mathbf{H}(x) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i) = \sum_{\alpha=n+1}^N H_\alpha(x) e_\alpha, \quad H_\alpha(x) := \frac{1}{n} \sum_{i=1}^n h_\alpha(e_i, e_i).$$

It is well known that the necessary and sufficient conditions for M in \mathbb{E}^N to be biharmonic, namely $\Delta \mathbf{H} = 0$, are the following (cf. [2, 3, 4]):

$$(2) \quad \begin{cases} \Delta^\perp \mathbf{H} - \sum_{i=1}^n h(A_{\mathbf{H}} e_i, e_i) = 0, \\ n \nabla |\mathbf{H}|^2 + 4 \operatorname{trace} A_{\nabla^\perp \mathbf{H}} = 0, \end{cases}$$

where Δ^\perp is the (non-positive) Laplace operator associated with the normal connection ∇^\perp . Similarly, the necessary and sufficient condition for M in \mathbb{E}^N to be biminimal is the following:

$$(3) \quad \Delta^\perp \mathbf{H} - \sum_{i=1}^n h(A_{\mathbf{H}} e_i, e_i) = \lambda \mathbf{H}.$$

3. NON EXISTENCE THEOREM FOR BIMINIMAL SUBMANIFOLD

In this section, we show that a *nonnegative* biminimal properly immersed submanifold (that is, a biminimal properly immersed submanifold with $\lambda \geq 0$) in the Euclidean space is minimal.

From the equation of (3), we have the following.

Lemma 3.1. *Let $M = (M, g)$ be a nonnegative biminimal submanifold in \mathbb{E}^N . Then, the following inequality for $|\mathbf{H}|^2$ holds*

$$(4) \quad \Delta |\mathbf{H}|^2 \geq \frac{2}{n} |\mathbf{H}|^4.$$

Proof. The equation of (3) implies that, at each $x \in M$,

$$\begin{aligned}
 \Delta |\mathbf{H}|^2 &= 2 \sum_{i=1}^n \langle \nabla_{e_i}^\perp \mathbf{H}, \nabla_{e_i}^\perp \mathbf{H} \rangle + 2 \langle \Delta^\perp \mathbf{H}, \mathbf{H} \rangle \\
 (5) \quad &= 2 \sum_{i=1}^n \langle \nabla_{e_i}^\perp \mathbf{H}, \nabla_{e_i}^\perp \mathbf{H} \rangle + 2 \sum_{i=1}^n \langle h(A_{\mathbf{H}} e_i, e_i), \mathbf{H} \rangle + 2 \lambda \langle \mathbf{H}, \mathbf{H} \rangle \\
 &\geq 2 \sum_{i=1}^n \langle A_{\mathbf{H}} e_i, A_{\mathbf{H}} e_i \rangle.
 \end{aligned}$$

When $\mathbf{H}(x) \neq 0$, set $e_N := \frac{\mathbf{H}(x)}{|\mathbf{H}(x)|}$. Then, $\mathbf{H}(x) = H_N(x)e_N$ and $|\mathbf{H}(x)|^2 = H_N(x)^2$. From (6), we have at x

$$\begin{aligned}
 \Delta |\mathbf{H}|^2 &\geq 2 H_N^2 \sum_{i=1}^n \langle A_{e_N} e_i, A_{e_N} e_i \rangle \\
 (6) \quad &= 2 |\mathbf{H}|^2 |h_N|_g^2 \\
 &\geq \frac{2}{n} |\mathbf{H}|^4.
 \end{aligned}$$

Even when $\mathbf{H}(x) = 0$, the above inequality (4) still holds at x . This completes the proof. \square

Theorem 3.2. *Any nonnegative biminimal properly immersed submanifold in \mathbb{E}^N is minimal.*

Proof. If M is compact, applying the standard maximum principle to the elliptic inequality (4), we have that $\mathbf{H} = 0$ on M . Therefore, we may assume that M is noncompact. Suppose that $\mathbf{H}(x_0) \neq 0$ at some point $x_0 \in M$. Then, we will lead a contradiction.

Set

$$u(x) := |\mathbf{H}(x)|^2 \quad \text{for } x \in M.$$

For each $\rho > 0$, consider the function

$$F(x) = F_\rho(x) := (\rho^2 - |\mathbf{x}(x)|^2)^2 u(x) \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho}).$$

Then, there exists $\rho_0 > 0$ such that $x_0 \in \mathbf{x}^{-1}(\mathbf{B}_{\rho_0})$. For each $\rho \geq \rho_0$, $F = F_\rho$ is a nonnegative function which is not identically zero on $M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho})$. Take any $\rho \geq \rho_0$ and fix it. Since M is properly immersed in \mathbb{E}^N , $M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho})$ is compact. By this fact combined with $F = 0$ on $M \cap \mathbf{x}^{-1}(\partial \overline{\mathbf{B}_\rho})$, there exists a maximum point $p \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho)$ of $F = F_\rho$ such that $F(p) > 0$. We have $\nabla F = 0$ at p , and hence

$$(7) \quad \frac{\nabla u}{u} = \frac{2 \nabla |\mathbf{x}(x)|^2}{\rho^2 - |\mathbf{x}(x)|^2} \quad \text{at } p.$$

We also have that $\Delta F \leq 0$ at p . Combining this with (7), we obtain

$$(8) \quad \frac{\Delta u}{u} \leq \frac{6 |\nabla |\mathbf{x}(x)|^2|_g^2}{(\rho^2 - |\mathbf{x}(x)|^2)^2} + \frac{2 \Delta |\mathbf{x}(x)|^2}{\rho^2 - |\mathbf{x}(x)|^2} \quad \text{at } p.$$

From $\Delta \mathbf{x} = n\mathbf{H}$, we note

$$(9) \quad \begin{cases} \Delta |\mathbf{x}(x)|^2 = 2 \sum_{i=1}^n |\nabla_{e_i} \mathbf{x}(x)|^2 + 2 \langle \Delta \mathbf{x}(x), \mathbf{x}(x) \rangle \leq 2n + 2n|\mathbf{H}| \cdot |\mathbf{x}(x)|, \\ |\nabla |\mathbf{x}(x)|^2|_g^2 \leq 4n|\mathbf{x}(x)|^2. \end{cases}$$

It then follows from (4), (8) and (9) that

$$u(p) \leq \frac{12n^2 |\mathbf{x}(p)|^2}{(\rho^2 - |\mathbf{x}(p)|^2)^2} + \frac{2n^2(1 + \sqrt{u(p)}|\mathbf{x}(p)|)}{\rho^2 - |\mathbf{x}(p)|^2},$$

and hence

$$F(p) \leq 12n^2|\mathbf{x}(p)|^2 + 2n^2(\rho^2 - |\mathbf{x}(p)|^2) + 2n^2\sqrt{F(p)}|\mathbf{x}(p)|.$$

Therefore, there exists a positive constant $c(n) > 0$ depending only on n such that

$$F(p) \leq c(n)\rho^2.$$

Since $F(p)$ is the maximum of $F = F_\rho$, we have

$$F(x) \leq F(p) \leq c(n)\rho^2 \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho}),$$

and hence

$$(10) \quad |\mathbf{H}(x)|^2 = u(x) \leq \frac{c(n)\rho^2}{(\rho^2 - |\mathbf{x}(x)|^2)^2} \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho) \quad \text{and } \rho \geq \rho_0.$$

Letting $\rho \nearrow \infty$ in (10) for $x = x_0$, we have that

$$|\mathbf{H}(x_0)|^2 = 0.$$

This contradicts our assumption that $\mathbf{H}(x_0) \neq 0$. Therefore, M is minimal. \square

Especially, any free biminimal properly immersed submanifold in \mathbb{E}^N is minimal. From the equations (2), we have:

Corollary 3.3 ([1]). *Any biharmonic properly immersed submanifold in \mathbb{E}^N is minimal.*

This corollary gives an affirmative partial answer to Chen's conjecture.

4. BIMINIMAL SUBMANIFOLD WITH $\lambda < 0$

For the case of $\lambda < 0$, we shall construct biminimal submanifolds.

Proposition 4.1 ([13]). *Let $\phi : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion with $\mathbf{H} = H_{n+1}e_{n+1}$ its mean curvature vector. Then M is biminimal if and only if*

$$\Delta H_{n+1} = (|h|^2 + \lambda)H_{n+1},$$

for some value of λ in \mathbb{R} .

From this proposition, if M is a non-trivial biminimal submanifold with harmonic mean curvature, then $\lambda < 0$.

Corollary 4.2. *Let $\phi : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion with harmonic mean curvature. If M is free biminimal, then it is minimal.*

Corollary 4.3. *Let $\phi : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometric immersion with harmonic mean curvature. Then M is non-trivial biminimal if and only if*

$$|h|^2 = -\lambda,$$

for $\lambda(< 0) \in \mathbb{R}$.

Using this result, we obtain following:

Proposition 4.4. *The isometric immersion $\phi : S^n \left(\sqrt{\frac{n}{-\lambda}} \right) \rightarrow \mathbb{E}^{n+1}$, ($\lambda < 0$) is non-trivial biminimal.*

Proof. In this case, $A = -\frac{1}{\sqrt{\frac{n}{-\lambda}}}I$, where I is the identity transformation. Therefore, we have $|h|^2 = n\frac{1}{\sqrt{\frac{n}{-\lambda}}^2} = -\lambda$. \square

For the curve case, we shall construct biminimal curves.

Definition 4.5 ([13]). The Frenet frame $\{B_i\}_{i=1,\dots,N}$ associated with a curve $\gamma : I \subset \mathbb{R} \rightarrow (\tilde{M}, \langle \cdot, \cdot \rangle)$ is the orthonormalization of the $(N+1)$ -tuple $\left\{ \nabla_{\frac{\partial}{\partial t}}^{\gamma(k)} d\gamma \left(\frac{\partial}{\partial t} \right) \right\}_{k=0,\dots,N}$ described by

$$\begin{aligned} B_1 &= d\gamma \left(\frac{\partial}{\partial t} \right), \\ \nabla_{\frac{\partial}{\partial t}}^{\gamma} B_1 &= k_1 B_2, \\ \nabla_{\frac{\partial}{\partial t}}^{\gamma} B_i &= -k_{i-1} B_{i-1} + k_i B_{i+1} \quad \forall i = 2, \dots, N-1, \\ \nabla_{\frac{\partial}{\partial t}}^{\gamma} B_N &= -k_{N-1} B_{N-1}, \end{aligned}$$

where the functions $\{k_1 > 0, k_2, k_3, \dots, k_{N-1}\}$ are called the curvatures of γ . Note that $B_1 = \gamma'$ is the unit tangent vector field to the curve.

Biminimal curves in a Euclidean space are characterized as follows.

Proposition 4.6 ([13]). *Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^N$, $N \geq 2$, be a curve parametrized by arc length from an open interval of \mathbb{R} into a Euclidean space \mathbb{E}^N . Then γ is biminimal if and only if k_i fulfill the system:*

$$(11) \quad \begin{cases} k_1'' - k_1^3 - k_1 k_2^2 - \lambda k_1 = 0, \\ k_1^2 k_2 = \text{constant}, \\ k_1 k_2 k_3 = 0. \end{cases}$$

When $\lambda < 0$, using this proposition, we construct an example of biminimal curves.

Example. We consider the curve

$$\gamma(s) = \frac{1}{\sqrt{-\lambda}} \left\{ \cos(\sqrt{-\lambda}s) c_1 + \sin(\sqrt{-\lambda}s) c_2 \right\} + c_3, \quad (\lambda < 0),$$

where, c_1, c_2 are constant vectors orthogonal to each other with $|c_1|^2 = |c_2|^2 = 1$, and c_3 is a constant vector. Direct computation shows that the curve is non-trivial biminimal.

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